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APPROXIMATING COMMON FIXED POINTS OF FINITE FAMILY OF NONEXPANSIVE NONSELF-MAPPINGS IN A BANACH SPACE

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Abstract

In this paper, a new multi-step iterative scheme with errors involving a finite family of nonexpansive nonself-mappings in a Banach space is defined. Weak and strong convergence theorems of the new iterative scheme are established in a uniformly convex Banach space.

1. Introduction and Preliminaries

Let X be a real normed linear space and C a nonempty subset of X. A mapping $T : C \to C$ is said to be nonexpansive on C if

$$\| Tx - Ty \| \leq \| x - y \|$$

for all $x, y \in C$.

Fixed-point iteration process for nonexpansive mappings in Banach

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spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve the nonlinear operator equations in Hilbert spaces and Banach spaces; see [4, 8, 13, 14, 20, 21]. In 1993, Tan and Xu [20] introduced and studied a modified Ishikawa iteration process to approximate fixed points of non-expansive mappings defined on nonempty closed convex bounded subsets of a uniformly convex Banach space. Five years latter, Xu [25] introduced the iterative schemes known as Mann iterative scheme with errors and Ishikawa iterative scheme with errors. In 1998, Takahashi and Tamura [19] introduced and studied a generalization of Ishikawa iterative schemes for a pair of nonexpansive mappings in Banach spaces. In 2005, Khan and Fukhar-ud-din [7] extended their scheme to the modified Ishikawa iterative schemes with errors for two mappings and gave weak and strong convergence theorems. Iterative techniques for approximating fixed points of non-expansive nonself-mappings have been studied by various authors; see [24, 5, 18, 9]. In [16], Shahzad extended Tan and Xu's results [20, Theorem 1, p. 305] to the case of nonexpansive nonself-mapping in a uniformly convex Banach space. In 2006, Plubtieng and Ungchittrakool [12] extended the two-step iterative schemes defined by Shahzad [16] to the multi-step iterative scheme with errors for a finite family of nonexpansive nonself-mappings. They gave some weak and strong convergence theorems of such iterations for a finite family of nonexpansive nonself-mappings in uniformly convex Banach spaces. Recently, Thianwan and Suantai [22] introduced and studied the new class of three-step iterative scheme with errors for nonexpansive nonself-mappings and gave some strong and weak convergence theorems for such mappings.

Motivating these facts, a new multi-step iterative scheme with errors for a finite family of nonexpansive nonself-mappings is introduced and studied. Our schemes can be viewed as an extension for three-step iterative schemes of Thianwan and Suantai [22]. The scheme is defined as follows:

Let X be a normed space, C a nonempty convex subset of X, $P : X \to C$ a nonexpansive retraction of X onto C and $T_1, T_2, \dots, T_N : C \to X$ are given mappings. Then for an arbitrary $x_1 \in C$, the following iteration scheme is studied:

$$x_n^1 = P(\alpha_n^1 T_1 x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1)$$

$$x_n^2 = P(\alpha_n^2 T_2 x_n + \beta_n^2 x_n + \gamma_n^2 u_n^2)$$

$$\vdots \qquad \vdots$$

$$x_{n+1} = x_n^N = P(\alpha_n^N T_N x_n + \beta_n^N x_n^{N-1} + \gamma_n^N u_n^N), \tag{1.1}$$

$n \geq 1$, where $\{\alpha_n^1\}, \{\alpha_n^2\}, \ldots, \{\alpha_n^N\}, \{\beta_n^1\}, \{\beta_n^2\}, \ldots, \{\beta_n^N\}, \{\gamma_n^1\}, \{\gamma_n^2\}, \ldots,$ $\{\gamma_n^N\}$ are appropriate sequences in $[0, 1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for all $i = 1, 2, \ldots, N$, and $\{u_n^1\}, \{u_n^2\}, \ldots, \{u_n^N\}$ are bounded sequences in C.

If $N = 3$, $T_1 = T_2 = T_3 \equiv T$, $a_n = \alpha_n^1$, $c_n = \alpha_n^2$, $\alpha_n = \alpha_n^3$, $b_n = \gamma_n^1$, $d_n = \gamma_n^2$, $\beta_n = \gamma_n^3$, $u_n = u_n^1$, $v_n = u_n^2$, $w_n = u_n^3$, $z_n = x_n^1$ and $y_n = x_n^2$, then the iterative scheme (1.1) reduces to the iterative scheme with errors for a mapping defined by Thianwan and Suantai [22]:

$$z_n = P((1 - a_n - b_n)x_n + a_n T x_n + b_n u_n),$$

$$y_n = P((1 - c_n - d_n)z_n + c_n T x_n + d_n v_n),$$

$$x_{n+1} = P((1 - \alpha_n - \beta_n)y_n + \alpha_n T x_n + \beta_n w_n), \; n \geq 1, \tag{1.2}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in C and $\{a_n\}, \{b_n\}, \{c_n\},$ $\{d_n\}, \{\alpha_n\}, \{\beta_n\}$ are appropriate sequences in $[0, 1]$.

The purpose of this paper is to establish several strong and weak convergence theorems of the multi-step iterative scheme with errors (1.1) for a finite family of nonexpansive nonself-mappings in a uniformly convex Banach space. More precisely, we prove weak convergence of the iteration process in a uniformly convex Banach space X such that its dual X^* has the Kadee-Klee property.

Now, we recall the well known concepts and results.

Let X be a Banach space with dimension $X \geq 2$. The modulus of X is the function $\delta_X : (0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\| \frac{1}{2}(x + y)\right\| : \| x \| = 1, \| y \| = 1, \varepsilon = \| x - y \|\right\}.$$

Banach space X is uniformly convex if and only if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.

A subset C of X is said to be retract if there exists continuous mapping $P : X \to C$ such that $Px = x$ for all $x \in C$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : X \to X$ is said to be a retraction if $P^2 = P$. If a mapping P is a retraction, then $Pz = z$ for every $z \in R(P)$, range of P. A mapping $T : C \to X$ is called *demi-closed* with respect to $y \in X$ if for each sequence $\{x_n\}$ in C and each $x \in X$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Recall that a Banach space X is said to satisfy *Opial's condition* [11] if $x_n \to x$ weakly as $n \to \infty$ and $x \neq y$ implying that

$$\limsup_{n\to\infty} \| x_n - x \| < \limsup_{n\to\infty} \| x_n - y \|.$$

A Banach space X is said to have the Kadec-Klee property if for every sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ and $\| x_n \| \to \| x \|$ together imply $\| x_n - x\| \to 0$. A family $\{T_i : i = 1, 2, ..., N\}$ of N nonself-mappings of C (i.e., $T_i : C \to C$) with $F = \bigcap_{i=1}^{N} F(T_i) \neq \varnothing$ is said to satisfy condition (B) on C [3] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\max_{1 \le l \le N}\{\| x - T_l x \|\} \ge f(d(x, F)) \tag{1.3}$$

for all $x \in C$; see [15, p. 377] for an example of nonexpansive mappings satisfying condition (B). The family $\{T_i : i = 1, 2, ..., N\}$ is said to satisfy condition (A) if (1.3) is replaced by

$$\| x - Tx \| \ge f(d(x, F(T))).$$

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1.1 ([20]). *Let* $\{s_n\}, \{t_n\}$ *be two nonnegative sequences satisfying*

$$s_{n+1} \leq s_n + t_n \text{ for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, *then* $\lim_{n\to\infty} s_n$ *exists. Moreover, if there exists a subsequence* $\{s_{n_j}\}$ *of* $\{s_n\}$ *such that* $s_{n_j} \to 0$ *as* $j \to \infty$, *then* $s_n \to 0$ *as* $n \to \infty$.

Lemma 1.2 ([23]). *Let* $p > 1$ *and* $R > 1$ *be two fixed numbers and X a Banach space. Then X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function* $g : [0, \infty) \to [0, \infty)$ *with* $g(0) = 0$ *such that* $\| \lambda x + (1-\lambda) y \|^p \leq \lambda \| x \|^p + (1-\lambda) \| y \|^p - W_p(\lambda) g(\| x - y \|)$ *for all* $x, y \in B_R(0) = \{x \in X : \| x \| \leq R\}$, *and* $\lambda \in [0, 1]$, *where* $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

Lemma 1.3 ([6]). *Let X be a real reflexive Banach space such that its dual* X^* *has the Kadec-Klee property. Let* $\{x_n\}$ *be a bounded sequence in* X *and* $x^*, y^* \in \omega_w(x_n)$; *here* $\omega_w(x_n)$ *denote the set of all weak subsequential limits of* $\{x_n\}$. *Suppose* $\lim_{n\to\infty} \| t x_n + (1-t) x^* - y^* \|$ *exists for all* $t \in [0, 1]$. *Then* $x^* = y^*$.

Lemma 1.4 ([1]). *Let X be a uniformly convex Banach space, C a nonempty closed convex subset of X and* $T : C \to X$ *a nonexpansive mapping. Then* $I - T$ *is demi-closed at zero, i.e., if* $x_n \to x$ *weakly and* $x_n - Tx_n \to 0$ *strongly, then* $x \in F(T)$, *where* $F(T)$ *is the set of fixed points of T*:

Lemma 1.5 ([17]). *Let X be a Banach space which satisfies Opial's condition and let* $\{x_n\}$ *be a sequence in X. Let* $u, v \in X$ *be such that* $\lim_{n\to\infty} \| x_n - u \|$ *and* $\lim_{n\to\infty} \| x_n - v \|$ *exist. If* $\{x_{n_k}\}$ *and* $\{x_{m_k}\}$ *are subsequences of* $\{x_n\}$ *which converge weakly to u and v, respectively, then* $u = v$.

We denote by Γ the set of strictly increasing, continuous convex function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ with $\gamma(0) = 0$. Let C be a convex subset of the Banach space X. A mapping $T : C \to C$ is said to be type (γ) if $\gamma \in \Gamma$ and $0 \leq \alpha \leq 1$,

$$\gamma(\| \alpha Tx + (1 - \alpha)Ty - T(\alpha x + (1 - \alpha)y) \|) \leq \| x - y \| - \| Tx - Ty \|$$

for all x, y in C.

Lemma 1.6 ([2], [10]). *Let X be a uniformly convex Banach space and C a convex subset of X. Then there exists $\gamma \in \Gamma$ such that for each mapping $S : C \to C$ with Lipschitz constant L,*

$$\| \alpha Sx + (1 - \alpha)Sy - S(\alpha x + (1 - \alpha)y) \| \leq L\gamma^{-1}\left(\| x - y \| - \frac{1}{L}\| Sx - Sy \|\right)$$

for all x, $y \in C$ and $0 < \alpha < 1$.

2. Main Results

In this section, we prove weak and strong convergence theorems of the iterative scheme given in (1.1) to a common fixed point for a finite family of nonexpansive nonself-mappings in a uniformly convex Banach space. In order to prove our main results, the following lemmas are needed.

Lemma 2.1. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be nonexpansive nonself-mappings. Let $\{x_n\}$ be the sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ for each $i = 1, 2, \ldots, N$. If $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in F$.*

Proof. Let $x^* \in F$. Using (1.1), for each $n \geq 1$, we have

$$\|x_n^1 - x^*\| = \|P(\alpha_n^1 T_1 x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1) - x^*\|$$

$$
\begin{aligned}
&= \|P(\alpha_n^1 T_1 x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1) - P(x^*)\| \\
&\le \|\alpha_n^1 T_1 x_n + \beta_n^1 x_n + \gamma_n^1 u_n^1 - x^*\| \\
&\le \alpha_n^1 \|T_1 x_n - x^*\| + \beta_n^1 \|x_n - x^*\| + \gamma_n^1 \|u_n^1 - x^*\| \\
&\le \alpha_n^1 \|x_n - x^*\| + \beta_n^1 \|x_n - x^*\| + \gamma_n^1 \|u_n^1 - x^*\| \\
&\le \|x_n - x^*\| + d_n^1,
\end{aligned}
$$

where $d_n^1 = \gamma_n^1 \|u_n^1 - x^*\|$. Since $\sum_{n=1}^{\infty} \gamma_n^1 < \infty$, $\sum_{n=1}^{\infty} d_n^1 < \infty$. Next, we note that

$$
\begin{aligned}
\|x_n^2 - x^*\| &= \|P(\alpha_n^2 T_2 x_n + \beta_n^2 x_n^1 + \gamma_n^2 u_n^2) - x^*\| \\
&= \|P(\alpha_n^2 T_2 x_n + \beta_n^2 x_n^1 + \gamma_n^2 u_n^2) - P(x^*)\| \\
&\le \|\alpha_n^2 T_2 x_n + \beta_n^2 x_n^1 + \gamma_n^2 u_n^2 - x^*\| \\
&\le \alpha_n^2 \|T_2 x_n - x^*\| + \beta_n^2 \|x_n^1 - x^*\| + \gamma_n^2 \|u_n^2 - x^*\| \\
&\le \alpha_n^2 \|x_n - x^*\| + \beta_n^2 \|x_n^1 - x^*\| + \gamma_n^2 \|u_n^2 - x^*\| \\
&\le \alpha_n^2 \|x_n - x^*\| + \beta_n^2 (\|x_n - x^*\| + d_n^1) + \gamma_n^2 \|u_n^2 - x^*\| \\
&= \alpha_n^2 \|x_n - x^*\| + \beta_n^2 \|x_n - x^*\| + \beta_n^2 d_n^1 + \gamma_n^2 \|u_n^2 - x^*\| \\
&= (\alpha_n^2 + \beta_n^2) \|x_n - x^*\| + \beta_n^2 d_n^1 + \gamma_n^2 \|u_n^2 - x^*\| \\
&\le \|x_n - x^*\| + d_n^2,
\end{aligned}
$$

where $d_n^2 = \beta_n^2 d_n^1 + \gamma_n^2 \|u_n^2 - x^*\|$. Since $\sum_{n=1}^{\infty} d_n^1 < \infty$ and $\sum_{n=1}^{\infty} \gamma_n^2 < \infty$, $\sum_{n=1}^{\infty} d_n^2 < \infty$. Similarly, we have

$$
\begin{aligned}
\|x_n^3 - x^*\| &= \|P(\alpha_n^3 T_3 x_n + \beta_n^3 x_n^2 + \gamma_n^3 u_n^3) - x^*\| \\
&= \|P(\alpha_n^3 T_3 x_n + \beta_n^3 x_n^2 + \gamma_n^3 u_n^3) - P(x^*)\| \\
&\le \|\alpha_n^3 T_3 x_n + \beta_n^3 x_n^2 + \gamma_n^3 u_n^3 - x^*\|
\end{aligned}
$$

$$\leq \alpha_n^3\|T_3x_n - x^*\| + \beta_n^3\|x_n^2 - x^*\| + \gamma_n^3\|u_n^3 - x^*\|$$

$$\leq \alpha_n^3\|x_n - x^*\| + \beta_n^3(\|x_n - x^*\| + d_n^2) + \gamma_n^3\|u_n^3 - x^*\|$$

$$= \alpha_n^3\|x_n - x^*\| + \beta_n^3\|x_n - x^*\| + \beta_n^3 d_n^2 + \gamma_n^3\|u_n^3 - x^*\|$$

$$= (\alpha_n^3 + \beta_n^3)\|x_n - x^*\| + \beta_n^3 d_n^2 + \gamma_n^3\|u_n^3 - x^*\|$$

$$\leq \|x_n - x^*\| + d_n^3,$$

where $d_n^3 = \beta_n^3 d_n^2 + \gamma_n^3\|u_n^3 - x^*\|$, and so $\sum_{n=1}^{\infty} d_n^3 < \infty$.

By continuing a similar method, there exists a nonnegative real sequences $\{d_n^i\}$ such that $\sum_{n=1}^{\infty} d_n^i < \infty$ and

$$\|x_n^i - x^*\| \leq \|x_n - x^*\| + d_n^i,\ \forall n \geq 1,\ \forall i = 1, 2, \dots, N. \tag{2.1}$$

Thus, by (2.1), we have $\|x_{n+1} - x^*\| = \|x_n^N - x^*\| \leq \|x_n - x^*\| + d_n^N$ for all $n \in \mathbb{N}$. Hence, by Lemma 1.1, $\lim_{n\to\infty}\|x_n - x^*\|$ exists. The proof is completed. $\square$

Lemma 2.2. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \dots, T_N : C \to X$ be nonexpansive nonself-mappings. Let $\{x_n\}$ be the sequence defined by* (1.1) *with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \dots, N$ for some $\varepsilon \in (0, 1)$. If $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $\limsup_{n\to\infty}(\alpha_n^i + \beta_n^i) < 1$ for all $i = 1, 2, ..., N$, then $\lim_{n\to\infty}\|x_n - T_ix_n\| = 0$ for all $i = 1, 2, \dots, N$.*

Proof. Let $x^* \in F$. Then, by Lemma 2.1, $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Let $\lim_{n\to\infty}\|x_n - x^*\| = r$. If $r = 0$, then by the continuity of each T_i the conclusion follows. Suppose that $r > 0$. Since $\{x_n\}$ and $\{u_n^i\}$ are bounded sequences for all $i = 1, 2, \ldots, N$, there exists $R > 0$ such that $x_n^{i-1} - x^* + \gamma_n^i(u_n^i - x_n^{i-1})$, $T_i x_n - x^* + \gamma_n^i(u_n^i - x_n^{i-1}) \in B_R(0)$ for all $n \geq 1$ and for all $i = 1, 2, \ldots, N$. Using Lemma 1.2 and (2.1), we have

$$\begin{aligned}
\|x_n^i - x^*\|^2 &= \|P(\alpha_n^i T_i x_n + \beta_n^i x_n^{i-1} + \gamma_n^i u_n^i) - x^*\|^2 \\
&\leq \|\alpha_n^i T_i x_n + \beta_n^i x_n^{i-1} + \gamma_n^i u_n^i - x^*\|^2 \\
&= \|\alpha_n^i (T_i x_n - x^* + \gamma_n^i (u_n^i - x_n^{i-1})) \\
&\quad + (1 - \alpha_n^i)(x_n^{i-1} - x^* + \gamma_n^i (u_n^i - x_n^{i-1}))\|^2 \\
&\leq \alpha_n^i \|T_i x_n - x^* + \gamma_n^i (u_n^i - x_n^{i-1})\|^2 \\
&\quad + (1 - \alpha_n^i)\|x_n^{i-1} - x^* + \gamma_n^i (u_n^i - x_n^{i-1})\|^2 \\
&\quad - W_2(\alpha_n^i) g(\|T_i x_n - x_n^{i-1}\|) \\
&\leq \alpha_n^i (\|x_n - x^*\| + \gamma_n^i \|u_n^i - x_n^{i-1}\|)^2 \\
&\quad + (1 - \alpha_n^i)(\|x_n^{i-1} - x^*\| + \gamma_n^i \|u_n^i - x_n^{i-1}\|)^2 \\
&\quad - W_2(\alpha_n^i) g(\|T_i x_n - x_n^{i-1}\|) \\
&\leq \alpha_n^i (\|x_n - x^*\| + d_n^{i-1} + \gamma_n^i \|u_n^i - x_n^{i-1}\|)^2 \\
&\quad + (1 - \alpha_n^i)(\|x_n - x^*\| + d_n^{i-1} + \gamma_n^i \|u_n^i - x_n^{i-1}\|)^2 \\
&\quad - W_2(\alpha_n^i) g(\|T_i x_n - x_n^{i-1}\|) \\
&= (\|x_n - x^*\| + \lambda_n^{i-1})^2 - W_2(\alpha_n^i) g(\|T_i x_n - x_n^{i-1}\|), \qquad (2.2)
\end{aligned}$$

where $\lambda_n^{i-1} := d_n^{i-1} + \gamma_n^i \|u_n^i - x_n^{i-1}\|$. Since $\sum_{n=1}^{\infty} d_n^{i-1} < \infty$, $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\|u_n^i - x_n^{i-1}\|\}$ is bounded, we have $\sum_{n=1}^{\infty} \lambda_n^{i-1} < \infty$. Since $\alpha_n^i \in [\varepsilon, 1 - \varepsilon]$ for some $\varepsilon \in (0, 1)$, it follows that $\lambda := \varepsilon^2 \leq W_2(\alpha_n^i)$ for all

$n \in N.$ This together with (2.2), we have

$$\lambda g(\|T_i x_n - x_n^{i-1}\|) \le (\|x_n - x^*\| + \lambda_n^{i-1})^2 - \|x_n^i - x^*\|^2$$
$$= \|x_n - x^*\|^2 + 2\lambda_n^{i-1}\|x_n - x^*\| + (\lambda_n^{i-1})^2$$
$$- \|x_{n+1} - x^*\|^2$$
$$= \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \rho_n^{i-1},$$

where $\rho_n^{i-1} := 2\lambda_n^{i-1}\|x_n - x^*\| + (\lambda_n^{i-1})^2$. Since $\sum_{n=1}^{\infty} \lambda_n^{i-1} < \infty$, we get $\sum_{n=1}^{\infty} \rho_n^{i-1} < \infty$. This implies that $\lim_{n\to\infty} g(\|T_i x_n - x_n^{i-1}\|) = 0$. Since g is strictly increasing and continuous at 0 with $g(0) = 0$, it follows that $\lim_{n\to\infty}\|T_i x_n - x_n^{i-1}\| = 0$ for all $i = 1, 2, \ldots, N$. Note that,

$$\|x_n^{i-1} - x_n^{i-2}\| = \|P(\alpha_n^{i-1}T_{i-1}x_n + \beta_n^{i-1}x_n^{i-2} + \gamma_n^{i-1}u_n^{i-1}) - x_n^{i-2}\|$$
$$= \|P(\alpha_n^{i-1}T_{i-1}x_n + \beta_n^{i-1}x_n^{i-2} + \gamma_n^{i-1}u_n^{i-1}) - P(x_n^{i-2})\|$$
$$\le \|\alpha_n^{i-1}(T_{i-1}x_n - x_n^{i-2}) + \gamma_n^{i-1}(u_n^{i-1} - x_n^{i-2})\|$$
$$\le \alpha_n^{i-1}\|T_{i-1}x_n - x_n^{i-2}\| + \gamma_n^{i-1}\|u_n^{i-1} - x_n^{i-2}\|.$$

Since $\lim_{n\to\infty}\|T_{i-1}x_n - x_n^{i-2}\| = 0$ and $\sum_{n=1}^{\infty} \gamma_n^{i-1} < \infty$, it follows that

$$\lim_{n\to\infty}\|x_n^{i-1} - x_n^{i-2}\| = 0.$$

Using (1.1), for all $i = 1, 2, \ldots, N$, we have

$$\|x_n^{i-1} - x_n\| = \|P(\alpha_n^{i-1}T_{i-1}x_n + \beta_n^{i-1}x_n^{i-2} + \gamma_n^{i-1}u_n^{i-1}) - x_n\|$$
$$\le \|\alpha_n^{i-1}T_{i-1}x_n + \beta_n^{i-1}x_n^{i-2} + \gamma_n^{i-1}u_n^{i-1} - x_n\|$$
$$= \|\alpha_n^{i-1}(T_{i-1}x_n - x_n) + \beta_n^{i-1}(x_n^{i-2} - x_n) + \gamma_n^{i-1}(u_n^{i-1} - x_n)\|$$

$$= \|\alpha_n^{i-1}(T_{i-1}x_n - x_n^{i-2} + x_n^{i-2} - x_n)$$

$$+ \beta_n^{i-1}(x_n^{i-2} - x_n) + \gamma_n^{i-1}(u_n^{i-1} - x_n)\|$$

$$\leq \alpha_n^{i-1}\|T_{i-1}x_n - x_n^{i-2}\| + \alpha_n^{i-1}\|x_n^{i-2} - x_n\|$$

$$+ \beta_n^{i-1}\|x_n^{i-2} - x_n\| + \gamma_n^{i-1}\|u_n^{i-1} - x_n\|$$

$$\leq \alpha_n^{i-1}\|T_{i-1}x_n - x_n^{i-2}\| + \alpha_n^{i-1}\|x_n^{i-2} - x_n^{i-1}$$

$$+ x_n^{i-1} - x_n\| + \beta_n^{i-1}\|x_n^{i-2} - x_n^{i-1} + x_n^{i-1} - x_n\|$$

$$+ \gamma_n^{i-1}\|u_n^{i-1} - x_n\|$$

$$\leq \alpha_n^{i-1}\|T_{i-1}x_n - x_n^{i-2}\| + \alpha_n^{i-1}\|x_n^{i-2} - x_n^{i-1}\|$$

$$+ \alpha_n^{i-1}\|x_n^{i-1} - x_n\| + \beta_n^{i-1}\|x_n^{i-2} - x_n^{i-1}\|$$

$$+ \beta_n^{i-1}\|x_n^{i-1} - x_n\| + \gamma_n^{i-1}\|u_n^{i-1} - x_n\|.$$

This implies that

$$(1 - \alpha_n^{i-1} - \beta_n^{i-1})\|x_n^{i-1} - x_n\| \leq \alpha_n^{i-1}\|T_{i-1}x_n - x_n^{i-2}\|$$

$$+ \alpha_n^{i-1}\|x_n^{i-2} - x_n^{i-1}\|$$

$$+ \beta_n^{i-1}\|x_n^{i-2} - x_n^{i-1}\|$$

$$+ \gamma_n^{i-1}\|x_n^{i-1} - x_n\|.$$

If $\limsup_{n\to\infty}(\alpha_n^i + \beta_n^i) < 1$ for all $i = 1, 2, \ldots, N$, then there exist a positive integer n_0 and $\eta \in (0, 1)$ such that $\alpha_n^i + \beta_n^i < \eta < 1$ for all $\eta \geq n_0$. Hence

$$(1 - \eta)\|x_n^{i-1} - x_n\| \leq \|T_{i-1}x_n - x_n^{i-2}\| + \gamma_n^{i-1}\|u_n^{i-1} - x_n\|$$

$$+ 2\|x_n^{i-2} - x_n^{i-1}\|, \forall n \geq n_0.$$

Since $\lim_{n\to\infty}\|T_{i-1}x_n - x_n^{i-2}\| = 0$, $\lim_{n\to\infty}\|x_n^{i-2} - x_n^{i-1}\| = 0$ and $\sum_{n=1}^{\infty}\gamma_n^{i-1} < \infty$, it follows that $\lim_{n\to\infty}\|x_n - x_n^{i-1}\| = 0$. Thus, for all $i = 1, 2, \ldots, N$, we have

$$\|x_n - T_i x_n\| = \|x_n - x_n^{i-1} + x_n^{i-1} - T_i x_n\|$$

$$\leq \|x_n - x_n^{i-1}\| + \|T_i x_n - x_n^{i-1}\| \to 0 \text{ as } n \to \infty.$$

The proof is completed. □

The following result gives a strong convergence for a finite family of nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Condition (B).

Theorem 2.3. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be nonexpansive nonself-mappings which are satisfying condition* (*B*). *Let $\{x_n\}$ be the sequence defined by* (1.1) with $\sum_{n=1}^{\infty}\gamma_n^i < \infty$ *and* $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ *for all* $i = 1, 2, \ldots, N$ *for some* $\varepsilon \in (0, 1)$. *If* $F := \bigcap_{n=1}^{N} F(T_i) \neq \emptyset$ *and* $\limsup_{n\to\infty}(\alpha_n^i + \beta_n^i) < 1$ *for all* $i = 1, 2, \ldots, N$, *then* $\{x_n\}$ *converges strongly to a common fixed point in F.*

Proof. By Lemma 2.2, $\lim_{n\to\infty}\|T_i x_n - x_n\| = 0$ for all $i = 1, 2, \ldots, N$. Now by condition (B), $f(d(x_n, F)) \leq M_n := \max_{1\leq i\leq N}\{\|T_i x_n - x_n\|\}$ for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F) = 0$. Now we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{y_j\} \in F$ such that $\|x_{n_j} - y_j\| < 2^{-j}$. By the following method of the proof of Tan and Xu [20], we get that $\{y_j\}$ is a Cauchy sequence in F and so it converges. Let

$y_j \to y$. Since F is closed, therefore $y \in F$ and then $x_{n_j} \to y$. By Lemma 2.1, $\lim_{n\to\infty}\|x_n - x^*\|$ exists for all $x^* \in F$, $x_n \to y \in F$. □

If $N = 3$, $T_1 = T_2 = T_3 \equiv T$, $\alpha_n = a_n^1$, $c_n = \alpha_n^2$, $\alpha_n = \alpha_n^3$, $b_n = \gamma_n^1$, $d_n = \gamma_n^2$, $\beta_n = \gamma_n^3$, $u_n = u_n^1$, $v_n = u_n^2$, $w_n = u_n^3$, $z_n = x_n^1$ and $y_n = x_n^2$, then the iterative scheme (1.1) reduces to that of (1.2) and the following result is obtained.

Theorem 2.4. *Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T : C \to X$ a nonexpansive nonself-mapping with $F(T) \neq \varnothing$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ are sequences of real numbers in $[0, 1]$ with $c_n + d_n \in [0, 1]$ and $\alpha_n + \beta_n \in [0, 1]$ for all $n \geq 1$, and $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$. Suppose that T satisfies condition (A): If*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$,

and

(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$, *and* $\limsup_{n\to\infty} a_n < 1$,

then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ defined by the three-step iterative scheme (1.2) *converge strongly to a fixed point of T.*

Proof. Let $x^* \in F(T)$. Then, as in the proof of [22, Lemma 2.1 (i)], $\{x_n\}$ is bounded, $\lim_{n\to\infty}\|x_n - x^*\|$ exists and

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| + M(b_n + d_n + \beta_n),$$

where $M_1 = \sup\{\|u_n - x^*\| : n \geq 1\}$, $M_2 = \sup\{\|v_n - x^*\| : n \geq 1\}$, $M_3 = \sup\{\|w_n - x^*\| : n \geq 1\}$, $M = \max\{M_i : i = 1, 2, 3\}$, and so $\sum_{n=1}^{\infty} M(b_n + d_n + \beta_n) < \infty$ for all $n \geq 1$. This implies that $d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + M(b_n + d_n + \beta_n)$ and so, by Lemma 1.1, $\lim_{n\to\infty} d(x_n,$

$F(T))$ exists. Also, by [22, Lemma 2.1 (iv)], $\lim_{n\to\infty}\| x_n - Tx_n \| = 0.$ Since T satisfies condition (A), we conclude that $\lim_{n\to\infty} d(x_n, F(T)) = 0.$ Next we show that $\{x_n\}$ is a Cauchy sequence.

Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty} M(b_n + d_n + \beta_n) < \infty,$ given any $\varepsilon < 0,$ there exists a natural number n_0 such that $d(x_n, F(T)) < \frac{\varepsilon}{4}$ and $\sum_{k=n_0}^{n} M(b_k + d_k + \beta_k) < \frac{\varepsilon}{2}$ for all $n \geq n_0$. So we can find $y^* \in F(T)$ such that $\|x_{n_0} - y^*\| < \frac{\varepsilon}{4}.$ For $n \geq n_0$ and $m \geq 1,$ we have

$$\| x_{n+m} - x_n \| = \|x_{n+m} - y^*\| + \|x_n - y^*\|$$

$$\leq \|x_{n_0} - y^*\| + \|x_{n_0} - y^*\| + \sum_{k=n_0}^{n} M(b_k + d_k + \beta_k)$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $\{x_n\}$ is a Cauchy sequence and so is convergent since X is complete. Let $\lim_{n\to\infty} x_n = u.$ Then $d(u, F(T)) = 0.$ It follows that $u \in F(T).$ This completes the proof.

Theorem 2.5. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be nonexpansive nonself-mappings. Suppose that one of the mappings in $\{T_i : i = 1, 2, \ldots, N\}$ is completely continuous. Let $\{x_n\}$ be the sequence defined by* (1.1) *with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1-\varepsilon]$ for all $i = 1, 2, \ldots, N$ for some $\varepsilon \in (0, 1)$. If $F := \bigcap_{i=1}^{N} F(T_i) = \varnothing$ and $\limsup_{n\to\infty}(\alpha_n^i + \beta_n^i) < 1$ for all $i = 1, 2, \ldots, N,$ then $\{x_n\}$ converges strongly to a common fixed point in F.*

Proof. By Lemma 2.1, $\{x_n\}$ is bounded. In addition, by Lemma 2.2, $\lim_{n\to\infty}\| x_n - T_i x_n \| = 0$ for all $i = 1, 2, \ldots, N$, then $\{T_i x_n\}$ are also bounded for all $i = 1, 2, \ldots, N$. If T_k is completely continuous for some $k \in \{1, 2, \ldots, N\}$, then there exists a subsequence $\{T_k x_{n_j}\}$ of $\{T_k x_n\}$ such that $T_k x_{n_j} \to x^*$ as $j \to \infty$. It follows from Lemma 2.2 that $\lim_{j\to\infty}\|x_{n_j} - T_k x_{n_j}\| = 0$. So by the continuity of T_k and Lemma 1.4, we have $\lim_{j\to\infty}\|x_{n_j} - x^*\| = 0$ and $x^* \in F$. Furthermore, by Lemma 2.1, we get that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Thus $\lim_{n\to\infty}\|x_n - x^*\| = 0$. The proof is completed.

For $N = 3$, $T_1 = T_2 = T_3 \equiv T$, $a_n = \alpha_n^1$, $c_n = \alpha_n^2$, $\alpha_n = \alpha_n^3$, $b_n = \gamma_n^1$, $d_n = \gamma_n^2$, $\beta_n = \gamma_n^3$, $u_n = u_n^1$, $v_n = u_n^2$, $w_n = u_n^3$, $z_n = x_n^1$ and $y_n = x_n^2$ in Theorem 2.5, we obtain the following result.

Corollary 2.6 ([22, Theorem 2.2])**.** *Let X be a uniformly convex Banach space, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and $T : C \to X$ a completely continuous nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}, \{a_n\}$ $\{b_n\}, \{c_n\}$ and $\{d_n\}$ are sequences of real numbers in $[0,1]$ with $c_n + d_n \in [0,1]$ and $\alpha_n + \beta_n \in [0, 1]$ for all $n \geq 1$, and $\sum_{n=1}^\infty b_n < \infty$, $\sum_{n=1}^\infty d_n < \infty$, $\sum_{n=1}^\infty \beta_n < \infty$, and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$, *and*

(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$ *and* $\limsup_{n\to\infty} \alpha_n < 1$.

Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be the sequences defined by the three-step iterative scheme (1.2)*. Then $\{x_n\}, \{y_n\}$ and $\{z_n\}$ converge strongly to a Fixed point of T.*

We recall that a mapping $T : C \to C$ is called semi-compact (or hemicompact) if any sequence $\{x_n\}$ in C satisfying $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ has a convergent subsequence.

Theorem 2.7. *Let X be a uniformly convex Banach space and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be nonexpansive nonself-mappings. Suppose that one of the mappings in $\{T_i : i = 1,2, \ldots, N\}$ is semi-compact. Let $\{x_n\}$ be the sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1 - \varepsilon]$ for all $i = 1, 2, \ldots, N$ for some $\varepsilon \in (0,1)$. If $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ and $\limsup_{n\to\infty}(\alpha_n^i + \beta_n^i) < 1$ for all $i = 1, 2, \ldots, N$, then $\{x_n\}$ converges strongly to a common fixed point in F.*

Proof. Suppose that T_{i_0} is semi-compact for some $i_0 \in \{1, 2, ..., N\}$. By Lemma 2.2, we have $\lim_{n\to\infty}\|x_n - T_{i_0}x_n\| = 0$. Since T_{i_0} is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $x^* \in C$ such that $x_{n_j} \to x^*$ as $j \to \infty$. Now Lemma 2.2 guarantees that $\lim_{j\to\infty}\|x_{n_j} - T_i x_{n_j}\| = 0$ for all $i = 1, 2, \ldots, N$. Hence $\|x^* - T_i x^*\| = 0$ for all $i = 1, 2, \ldots, N$. This implies that $x^* \in F$. By Lemma 2.1, $\lim_{n\to\infty}\|x_n - x^*\|$ exists and then $\lim_{n\to\infty}\|x_n - x^*\| = \lim_{j\to\infty}\|x_{n_j} - x^*\| = 0$. This completes the proof.

In the next result, we prove weak convergence of the sequence $\{x_n\}$ defined by (1.1) in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.8. *Let X be a uniformly convex Banach space which satisfies Opia's condition and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, ..., T_N : C \to X$ be nonexpansive nonself-mappings. Let $\{x_n\}$ be a sequence defined by (1.1) with $\sum_{n=1}^{\infty} \gamma_n^i < \infty$ and $\{\alpha_n^i\} \subseteq [\varepsilon, 1 - \varepsilon]$ for all $i = 1, 2, ..., N$ for* *some* $\varepsilon \in (0, 1)$. *If* $F := \bigcap_{i=1}^{N} F(T_i) = \varnothing$ *and* $\limsup_{n\to\infty}(\alpha_n^i + \beta_n^i) < 1$ *for all* $i = 1, 2, \ldots, N$, *then* $\{x_n\}$ *converges weakly to a common fixed point in F.*

Proof. Let $x^* \in F$. By Lemma 2.1, $\lim_{x\to\infty}\|x_n - x^*\|$ exists. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in *F*. To prove this, let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ and $z_1, z_2 \in C$ such that $x_{n_i} \to z_1$ weakly as $i \to \infty$ and $x_{n_j} \to z_2$ weakly as $j \to \infty$. By Lemma 2.2,

$$\lim_{i\to\infty}\|x_{n_i} - T_k x_{n_i}\| = 0 = \lim_{j\to\infty}\|x_{n_j} - T_k x_{n_j}\|$$

for all $k = 1, 2, \ldots, N$ and by Lemma 1.4 insures that $I - T_k$ are demi-closed at zero for all $k = 1, 2, \ldots, N$. Therefore we obtain $T_k z_1 = z_1$ and $T_k z_2 = z_2$ for all $k = 1, 2, \ldots, N$. Thus $z_1, z_2 \in F$. It follows from Lemma 1.5 that $z_1 = z_2$. Therefore $\{x_n\}$ converges weakly to a common fixed point in *F*.

For $N = 3$, $T_1 = T_2 = T_3 \equiv T$, $a_n = \alpha_n^1$, $c_n = \alpha_n^2$, $\alpha_n = \alpha_n^3$, $b_n = \gamma_n^1$, $d_n = \gamma_n^2$, $\beta = \gamma_n^3$, $u_n = u_n^1$, $v_n = u_n^2$, $w_n = u_n^3$, $z_n = x_n^1$ and $y_n = x_n^2$ in Theorem 2.8, we obtain the following result.

Corollary 2.9 ([22, Theorem 2.4]). *Let X be a uniformly convex Banach space which satisfies Opial's condition, C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction, and* $T : C \to X$ *a nonexpansive nonself-mapping with* $F(T) \neq \varnothing$. *Suppose that* $\{\alpha_n\}, \{\beta_n\}, \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}$ *are sequences of real numbers in* $[0, 1]$ *with* $c_n + d_n \in [0, 1]$ *and* $\alpha_n + \beta_n \in [0, 1]$ *for all* $n \geq 1$, *and* $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} \beta_n < \infty$, *and*

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n) < 1$,

and

(ii) $0 < \liminf_{n\to\infty} c_n \leq \limsup_{n\to\infty}(c_n + d_n) < 1$ *and*

$$\limsup_{n\to\infty} a_n < 1.$$

Let $\{x_n\}$ *be the sequence defined by three-step iterative scheme* (1.2). *Then* $\{x_n\}$ *converges weakly to a fixed point of* T.

Finally, we will prove weak convergence of the sequence $\{x_n\}$ defined by (1.1) in a uniformly convex Banach space X whose dual X^* has the Kadec-Klee property.

Theorem 2.10. *Let* X *be a uniformly convex Banach space and* C *a nonempty closed convex nonexpansive retract of* X *with* P *as a nonexpansive retraction. Let* $T_1, T_2, \ldots, T_N : C \to X$ *be nonexpansive nonself-mappings with* $F := \bigcap_{i=1}^N F(T_i) = \emptyset$. *From an arbitrary* $x_1 \in C$, *define the sequence* $\{x_n\}$ *by the iterative scheme* (1.1) *with* $\sum_{n=1}^\infty \gamma_n^i < \infty$ *for all* $i = 1, 2, \ldots, N$. *Then for all* $u, v \in F$, *the limit*

$$\lim_{n\to\infty}\|tx_n + (1-t)u - v\|$$

exists for all $t \in [0, 1]$.

Proof. It follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n - x^*\|$ exists for all $x^* \in F$. This implies that $\{x_n\}$ is bounded. Then there exists $R > 0$ such that $\{x_n\} \subset B_R(0) \cap C$. Let $a_n(t) := \| tx_n + (1-t)u - v \|$, where $t \in (0, 1)$. Then $\lim_{n\to\infty} a_n(0) = \| u - v \|$ and by Lemma 2.1, $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty}\| x_n - v \|$ exists. Without loss of the generality, we may assume that $\lim_{n\to\infty}\| x_n - u \| = r$ for some positive number r. For any $n \geq 1$ and for all $i = 1, 2, \ldots, N$, we define $A_n^i : C \to C$ by

$$A_n^i := P(\alpha_n^i T_i + \beta_n^i A_n^{i-1} + \gamma_n^i u_n^i),$$

where $A_n^0 = I$, the identity operator on C. Thus, for all $x, y \in C$, we have $\|A_n^i x - A_n^i y\| \le \alpha_n^i \| x - y \| + \beta_n^i \|A_n^{i-1} x - A_n^{i-1} y\|$ for all $i = 2, \ldots, N$, and $\|A_n^i x - A_n^1 y\| \le \alpha_n^1 \| x - y \| + \beta_n^1 \| x - y \| \le \| x - y \|$. This implies, by induction, that A_n^i is a nonexpansive mappping for all $i = 1, 2, \ldots, N$ and for all $n \in N$. Set $S_{n,m} := A_{n+m-1}^N A_{n+m-2}^N \cdots A_n^N$, n, $m \ge 1$ and $b_{n,m} := \|S_{n,m}(tx_n + (1-t)u) - (tS_{n,m}x_n + (1-t)S_{n,m}u)\|$, where $0 \le t \le 1$. It easy to see that $A_n^N x_n = x_{n+1}$, $S_{n,m}x_n = x_{n+m}$ and $\|S_{n,m}x - S_{n,m}y\| \le \| x - y \|$.

We show first that, for any $x^* \in F$, $\|S_{n,m}x^* - x^*\| \to 0$ uniformly for all $m \ge 1$ as $n \to \infty$. Indeed, for any $x^* \in F$, we have

$$\|A_n^i x^* - x^*\| \le \beta_n^i \|A_n^{i-1} x^* - x^*\| + \gamma_n^i \|u_n^i - x^*\|$$

for all $i = 2, \ldots, N$, and $\|A_n^1 x^* - x^*\| \le \gamma_n^1 \|u_n^1 - x^*\|$. Therefore

$$\|A_n^N x^* - x^*\| \le \sigma_n^2 \gamma_n^1 \|u_n^1 - x^*\| + \sigma_n^3 \gamma_n^2 \|u_n^2 - x^*\| + \cdots$$

$$+ \sigma_n^N \gamma_n^{N-1} \|u_n^{N-1} - x^*\| + \gamma_n^N \|u_n^N - x^*\| \le M \sum_{i=1}^N \gamma_n^i,$$

for all $n \ge 1$, where

$$M = \max\left\{\sup_{n\ge1}\{\|u_n^1 - x^*\|\}, \ldots, \sup_{n\ge1}\{\|u_n^N - x^*\|\}\right\} \text{ and } \sigma_n^k = \prod_{i=k}^N \beta_n^i.$$

Hence

$$\|S_{n,m}x^* - x^*\| \le \|A_{n+m-1}^N A_{n+m-2}^N \cdots A_n^N x^* - A_{n+m-1}^N A_{n+m-2}^N \cdots A_{n+1}^N x^*\|$$

$$+ \|A_{n+m-1}^N A_{n+m-2}^N \cdots A_{n+1}^N x^* - A_{n+m-1}^N A_{n+m-2}^N \cdots A_{n+2}^N x^*\|$$

$$\vdots \qquad \vdots$$

$$+ \| A^N_{n+m-1} A^N_{n+m-2} x^* - A^N_{n+m-1} x^* \| + \| A^N_{n+m-1} x^* - x^* \|$$

$$\leq \| A^N_n x^* - x^* \| + \| A^N_{n+1} x^* - x^* \| + \cdots + \| A^N_{n+m-1} x^* - x^* \|$$

$$\leq M \sum_{i=1}^{N} (\gamma^i_n + \gamma^i_{n+1} + \cdots + \gamma^i_{n+m-1})$$

$$\leq \delta^{x^*}_n,$$

where $\delta^{x^*}_n := M \sum_{i=1}^{\infty} \sum_{k=n}^{\infty} \gamma^i_k$. Since $\sum_{n=1}^{\infty} \gamma^i_n < \infty$ for all $i = 1, 2, \ldots, N$, we have $\delta^{x^*}_n \to 0$ as $n \to \infty$ and hence $\| S_{n,m} x^* - x^* \| \to 0$ as $n \to \infty$. It follows from Lemma 1.6 that

$$b_{n,m} = \| S_{n,m}(tx_n + (1-t)u) - (tS_{n,m} x_n + (1-t) S_{n,m} u) \|$$

$$\leq \gamma^{-1} (\| x_n - u \| - \| S_{n,m} x_n - S_{n,m} u \|)$$

$$= \gamma^{-1} (\| x_n - u \| - \| x_{n+m} - u + u - S_{n,m} u \|)$$

$$\leq \gamma^{-1} (\| x_n - u \| - \| x_{n+m} - u \| - \| S_{n,m} u - u \|).$$

Hence $\gamma(b_{n,m}) \leq (\| x_n - u \| - \| x_{n+m} - u \| - \| S_{n,m} u - u \|)$. This implies that $\lim_{n,m \to \infty} \gamma(b_{n,m}) = 0$. By the property of γ, we obtain that $\lim_{n,m \to \infty} b_{n,m} = 0$. Observe that

$$a_{n+m}(t) = \| tx_{n+m} + (1-t)u - v \|$$

$$= \| tS_{n,m} x_n + (1-t)u - v \|$$

$$\leq \| tS_{n,m} x_n + (1-t)u - S_{n,m}(tx_n + (1-t)u) \|$$

$$+ \| S_{n,m}(tx_n + (1-t)u) - v \|$$

$$= \| tS_{n,m} x_n + (1-t) S_{n,m} u - S_{n,m}(tx_n + (1-t)u) + (1-t)(u - S_{n,m} u) \|$$

$$+\|S_{n,m}(tx_n + (1-t)u) - v\|$$

$$\leq b_{n,m} + \|S_{n,m}(tx_n + (1-t)u) - v\| + (1-t)\|u - S_{n,m}u\|$$

$$\leq b_{n,m} + \|S_{n,m}(tx_n + (1-t)u) - S_{n,m}v\| + \|S_{n,m}v - v\|$$

$$+ (1-t)\|u - S_{n,m}u\|$$

$$\leq b_{n,m} + a_n(t) + \|S_{n,m}v - v\| + (1-t)\|u - S_{n,m}u\|$$

$$\leq b_{n,m} + a_n(t) + \delta_n^v + (1-t)\delta_n^u.$$

Consequently,

$$\limsup_{m\to\infty} a_m(t) = \limsup_{m\to\infty} a_{n+m}(t)$$

$$\leq \limsup_{m\to\infty}(b_{n,m} + a_n(t) + \delta_n^v + (1-t)\delta_n^u)$$

$$\leq \gamma^{-1}(\|x_n - u\| - (\lim_{m\to\infty}\| x_n - u \| - \delta_n^u))$$

$$+ a_n(t) + \delta_n^v + (1-t)\delta_n^u$$

and

$$\limsup_{n\to\infty} a_n(t) \leq \gamma^{-1}(0) + \liminf_{n\to\infty} a_n(t) + 0 + 0 = \liminf_{n\to\infty} a_n(t).$$

This implies that $\lim_{n\to\infty} a_n(t)$ exists for all $t \in [0, 1]$. This completes the proof. □

Theorem 2.11. *Let X be a real uniformly convex Banach space such that its dual X^* has the Kadec-Klee property and C a nonempty closed convex nonexpansive retract of X with P as a nonexpansive retraction. Let $T_1, T_2, \ldots, T_N : C \to X$ be a nonexpansive nonself-mappings with $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\limsup_{n\to\infty}(\alpha_n^i + \beta_n^i) < 1$. From arbitrary $x_1 \in C$ define the sequence $\{x_n\}$ by the iterative scheme* (1.1) *with $\sum_{n=1}^\infty \gamma_n^i < \infty$*

and $\alpha_n^i \in [\varepsilon, 1 - \varepsilon]$ *for all* $i = 1\ 2, \ldots, N$ *for some* $\varepsilon \in (0, 1)$. *Then* $\{x_n\}$ *converges weakly to some fixed point in F.*

Proof. It follows from Lemma 2.1 that the sequence $\{x_n\}$ is bounded. Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to a point $x^* \in C$. By Lemma 2.2, we have $\lim_{j\to\infty}\|x_{n_j} - T_i x_{n_j}\| = 0$ for all $i = 1, 2, \ldots, N$. Now using Lemma 1.4, we have $I - T_i$ is demi-closed at zero for all $i = 1, 2, \ldots, N$. This implies that $T_i x^* = x^*$ for all $i = 1, 2, \ldots, N$. Thus $x^* \in F$. Next we prove that $\{x_n\}$ converges weakly to x^*. Suppose that $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converging weakly to some y^*. Then $y^* \in C$ and so $x^*, y^* \in \omega_w(x_n) \cap F$. By Theorem 2.10,

$$\lim_{n\to\infty}\|tx_n + (1-t)x^* - y^*\|$$

exists for all $t \in [0, 1]$. It follows from Lemma 1.3, we have $x^* = y^*$. As a result, $\omega_w(x_n)$ is a singleton, and so $\{x_n\}$ converges weakly to some fixed point in F. □

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